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LETTER TO THE EDITOR

Dynamical renormalisation through classical equations of motion

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Abstract. Classical approximations for critical dynamics are taken as the basis of a new dynamical renormalisation group strategy in real space. The approach is phenomenological and avoids proliferation of interactions, as well as memory effects. For the square and cubic Glauber model, the best determinations of static critical properties come up with values for the dynamic exponent which appear compatible with the most recent estimates by Monte Carlo or series expansion methods.

The major difficulties encountered in a dynamical renormalisation approach to kinetic spin models on lattices are memory effects in coarse-grained equations of motion (Mazenko and Valls 1982) and possible dangerous consequences of truncations of the dynamical hierarchy (Indekeu and Stella 1980).

We introduce a new renormalisation scheme, which, due to its phenomenological character, remains free from these difficulties and allows us to compute directly the critical properties within the Markoffian dynamics. For the first time, we can complement, by a technically simple and physically transparent real-space renormalisation method, the numerical results on two- and three-dimensional kinetic Ising models provided by more difficult Monte Carlo (Jan and Stauffer 1982), Monte Carlo renormalisation (Tobochnik *et al* 1981, Katz *et al* 1982, Yalabik and Gunton 1982) and series expansion (Racz and Collins 1976) techniques.

We start from classical (mean-field) equations of motion for the order parameter in which the conventional self-consistency requirements are relaxed to allow non-classical dynamical scaling behaviour. We provide hereby the appropriate extension to dynamics of a static mean-field renormalisation approach, which has already been applied successfully to ordered (Indekeu *et al* 1982) and disordered (Droz *et al* 1982) spin models, and to geometrical phase transitions (De'Bell 1983, De'Bell and Lookman 1983).

Consider a kinetic Ising model on a d-dimensional hypercubic lattice with single spin-flip transitions. Close to equilibrium and to criticality, the magnetisation is expected to scale, for long times t, as

$$m(K, h, t) = L^{-d+y_H} m(K_c + L^{y_T}(K - K_c), L^{y_H} h, L^{-z} t), \tag{1}$$

where K is the reduced nearest-neighbour coupling with critical value K_c, h is the magnetic field, L a scale factor, y_T and y_H the static critical exponents and z the dynamic

exponent. Our strategy consists in defining a dynamical renormalisation mapping implicitly, through an approximate realisation of equation (1).

In the mean-field approximation for critical dynamics (Suzuki and Kubo 1968), a single spin σ_1 is surrounded by $2d$ nearest neighbours fixed to the average magnetisation $m(t)$. The classical equation of motion is then obtained by the self-consistency requirement that the average of σ_1, $m_1(t)$, equals $m(t)$. This implies the classical result $\Delta(=z/y_T)=1$, because in this approximation the inverse relaxation time is analytic in K at criticality.

Our proposal now is to surround σ_1 with $2d$ spins fixed to an 'effective magnetisation' $b_1(t)$ which is not self-consistently equated to $m_1(t)$, but which we ask to have the same scaling properties as the average magnetisation.

To explain our approach with a most simple example, we first write down the evolution equation of $m_1(t)$, for $h=0$ and linearised in $b_1(t)$ (since the magnetisation is infinitesimal in the scaling regime):

$$\mathrm{d}m_1(t)/\mathrm{d}t = -A_1(K)m_1(t) + B_1(K)b_1(t), \tag{2}$$

where $A_1(K)=1$ and $B_1(K)=2dK$. Next we consider a cluster of two interacting nearest-neighbour spins σ_1 and σ_2, each surrounded by $2d-1$ effective magnetisations $b_2(t)$. The equation of motion for $m_2(t)\equiv\frac{1}{2}\langle\sigma_1+\sigma_2\rangle(t)$ then reads

$$\mathrm{d}m_2(t)/\mathrm{d}t = -A_2(K)m_2(t) + B_2(K)b_2(t), \tag{3}$$

where $A_2(K)=1-\tanh K$ and $B_2(K)=(2d-1)K(1-\tanh^2 K)$. At this point, we define a renormalisation mapping of K into K', in the scaling regime, by imposing the following finite-size realisation of equation (1) on the long-time solutions of equations (2) and (3):

$$m_2(K, 0, t; [b_2(t)]) = L^{-d+y_H} m_1(K', 0, L^{-z}t; [b_1(L^{-z}t)]), \tag{4}$$

where $L=2^{1/d}$ is the length rescaling, and we assume that $b(t)$ scales like $m(t)$:

$$b_2(t) = L^{-d+y_H} b_1(L^{-z}t). \tag{5}$$

Taking the time-derivative of equation (4) and using equations (2), (3) and (5), we arrive at the following equations which determine the dynamic RG mapping:

$$A_1(K') = L^z A_2(K) \qquad B_1(K') = L^z B_2(K). \tag{6a, b}$$

With reference to the solutions of equations (2) and (3)

$$m_N(t) = m_N(0)\,\mathrm{e}^{-A_N t} + B_N\,\mathrm{e}^{-A_N t}\int_0^t \mathrm{e}^{A_N t'} b_N(t')\,\mathrm{d}t', \tag{7}$$

we notice that the map (6) is obtained by imposing the scalings (4) and (5) on the second terms. The first terms are transients (A_N^{-1} always finite) and are not involved in the scaling. Looking for a non-trivial fixed point $K=K'=K^*$ of equations (6*a*, *b*), we obtain an estimate for K_c ($=K^*$) and for the dynamic exponent z. Furthermore, taking the ratio of (6*a*) to (6*b*), we deduce a purely static RG map $K'(K)$ which is parametrically independent of z. This static recursion coincides with the one obtained with the static mean-field renormalisation approach (Indekeu *et al* 1982) and yields $y_T=\log(\mathrm{d}K'/\mathrm{d}K|_{K_c})/\log L$, and (upon inclusion of a magnetic field h) y_H.

The effective magnetisation $b(t)$ can be viewed as a symmetry-breaking boundary condition on systems of small sizes. Finite-size scaling (Fisher and Barber 1972) then suggests that greater accuracy can be obtained by increasing the number of sites N and N' $(=L^{-d}N)$ in the clusters. In table 1 we report the results of our mean-field renormalisation (MFRG) for the square lattice, using very small systems ($N \leq 4$).

Table 1. Results for the 2D square kinetic Ising model. The mean-field renormalisation (MFRG) and the reaction-field renormalisation (RFRG: the numbers between brackets) are compared with results of other methods. Standard mean-field gives $K_c = 0.25$ and $\Delta = 1$.

	N	N'	K_c	y_T	y_H	z	Δ
MFRG (RFRG)	2	1	0.347 (0.398)	0.60 (0.67)	1.41 (1.62)	1.17 (2.19)	1.95 (3.27)
	4	1	0.361 (0.402)	0.69 (0.79)	1.50 (1.67)	1.39 (2.14)	2.01 (2.71)
	4	2	0.370 (0.404)	0.78 (0.92)	1.58 (1.72)	1.60 (2.06)	2.05 (2.24)
Other methods			0.441[a]	1[a]	1.87[a]	2.2 ± 0.2[b]	2.2 ± 0.2[b]

[a] Exact.
[b] Jan and Stauffer (1982), Katz *et al* (1982).

For bigger systems ($N > 4$) $m_N(t)$ is no longer an 'eigenmode' of the dynamics, but couples to higher spin correlations. It is then necessary to analyse the eigenmodes $\Phi_{N,k}(t)$, $k = 1, 2, \ldots$, which satisfy equations of the form (2) or (3), with proper coefficients $A_{N,k}$ and $B_{N,k}$. When $b_N(t)$, which acts like an external driving force, is switched off, the kth mode relaxes with the intrinsic relaxation time $A_{N,k}^{-1}$. The slowest eigenmode, say $\Phi_{N,1}(t)$, is then the legitimate candidate to describe the critical relaxation of $m_N(t)$. More specifically, the appropriate equation that replaces equations (2) or (3) is obtained by taking the time derivative of the quantity $\Phi_{N,1}(t)$ multiplied by the equilibrium ratio $(m_N/\Phi_{N,1})_{eq}$. In this way, the dynamic recursions (6) again imply correctly the RG map of the statics (Indekeu *et al* 1982).

In table 2 we report our MFRG results for the simple cubic lattice ($d = 3$), using systems with $N \leq 8$, and compare them with series expansion analysis or field-theoretic predictions (Domb 1974, Le Guillou and Zinn-Justin 1980).

Table 2. Results for the 3D cubic kinetic Ising model. Our approaches MFRG and RFRG (numbers in brackets) are compared with results of other approximate methods. Mean-field gives $K_c = 0.167$ and $\Delta = 1$.

	N	N'	K_c	y_T	y_H	z	Δ
MFRG (RFRG)	2	1	0.203 (0.205)	0.65 (0.82)	1.89 (2.00)	0.97 (1.30)	1.48 (1.59)
	4	1	0.204 (0.207)	0.72 (0.91)	1.94 (2.06)	1.06 (1.43)	1.47 (1.57)
	4	2	0.205 (0.208)	0.79 (0.99)	1.98 (2.12)	1.15 (1.54)	1.46 (1.56)
	8	1	0.207 (0.209)	0.82 (1.01)	2.00 (2.13)	1.20 (1.55)	1.46 (1.54)
	8	2	0.208 (0.210)	0.90 (1.11)	2.05 (2.19)	1.31 (1.66)	1.46 (1.50)
	8	4	0.209 (0.212)	1.01 (1.22)	2.12 (2.26)	1.46 (1.79)	1.45 (1.46)
Other methods			0.222[a]	1.59[b]	2.48[b]	$\cong 2.0$[c]	$\cong 1.3$[c]

[a] Domb (1974).
[b] Le Guillou and Zinn-Justin (1980).
[c] De Dominicis *et al* (1975).

The MFRG results in tables 1 and 2 show a definite systematic improvment as the sizes increase. However, instead of proceeding towards bigger sizes, it is more convenient to ask if the results can improve by taking as a starting point of the renormalisation better classical approximations than standard mean-field ones. We were able to do this by starting from reaction-field approximations (Onsager 1936, Dekeyser and Halzen 1969) in thermal equilibrium, where the static effective magnetisation is made dependent on the spin configuration inside the cluster, such as to embody a reaction effect from the interior spins onto the driving forces at the boundary.

We introduce the 'effective reaction magnetisation' $r_1(\sigma_1)$ surrounding a single spin σ_1. This quantity is defined as the equilibrium average of a spin s, subject to the condition that one of its nearest neighbours is σ_1 and the others are fixed to b_1, the configuration-independent effective magnetisation. This kind of conditional averaging procedure is one way to derive Onsager's reaction-field correction to mean-field theory†. A straightforward calculation yields (linearising in b_1):

$$r_1(\sigma_1) = \sigma_1 \tanh K + (2d-1)K(1-\tanh^2 K)b_1.$$

The static magnetisation m_1 is now obtained as the equilibrium average of σ_1 surrounded by $2d$ spins fixed to $r_1(\sigma_1)$. In order to extend the reaction-field approximation to dynamics we replace b_1, just as before, by its time-dependent generalisation $b_1(t)$. The equation of motion for $m_1(t)$ is then given by equation (2) with $A_1(K) = 1$ and $B_1(K) = 2d(2d-1)K^2(1-\tanh^2 K)$.

In order to exemplify the generalisation of our procedure to larger clusters, we illustrate in figure 1 our reaction-field calculation for a cluster of two nearest-neighbour sites. Because of the anisotropic geometry we now construct, e.g., both a transverse reaction field $t_2(\sigma_1, \sigma_2)$ and a longitudinal field $r_2(\sigma_1)$ acting on the spin σ_1. Without going into technical details we mention that the effect of $t_2(\sigma_1, \sigma_2)$ is not only to modify the coupling of σ_1 and σ_2 to b_2, but also to change the interaction between σ_1 and σ_2. On the basis of the effective two-site Hamiltonian with modified interactions, the transition rates $w_j(\{\sigma\})$ are constructed according to the detailed-balance principle, with the standard choice $w_j(\{\sigma\}) = \frac{1}{2}\Pi_\alpha (1 - \sigma_j \tanh K_\alpha(K) \Sigma \sigma_\alpha)$, where the index α takes into account the possibility of reaction field generated interactions different from the nearest-neighbour one in more general cases.

In table 1 we present the results of our reaction-field renormalisation (RFRG) for the square lattice. When comparing RFRG and MFRG results in this table, it is clear that the inclusion of the reaction-field correction is working extremely well. For sizes

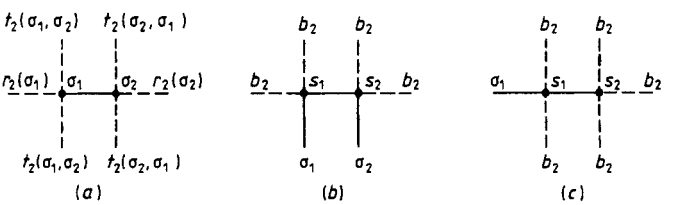

Figure 1. (*a*) A two-spin cluster is illustrated with its effective reaction-field boundary. (*b*) It is shown how to calculate, e.g., $t_2(\sigma_1, \sigma_2)$ as the cluster average over s_1 and s_2, of spin s_1. (*c*) The calculation of $r_2(\sigma_1)$, as the average over s_1 and s_2 of spin s_1, is shown.

† This was pointed out to us by A Maritan.

as small as $N = 4$ and $N' = 2$ we are now able to get all our estimates well within 10% of the exact results or possibly most reliable predictions. We draw attention to the fact that the estimate $z = 2.2(\pm 0.1)$ comes from a recent high-temperature analysis, a Monte Carlo simulation and Monte Carlo RG, and that older estimates were lower (typically 2.0 ± 0.1)†. Up to now, it has been extremely difficult, if not impossible, to draw reliable dynamical exponents from other real space RG methods (Indekeu and Stella 1980, Deker and Haake 1980, Mazenko and Valls 1981, Takano and Suzuki 1982, Haake and Lewenstein 1983).

In table 2 we report our RFRG results for the Glauber model on a cubic lattice.

Similarly to what happended in $d = 2$, the results are generally better now than in the MFRG case. Both MFRG and RFRG sequences for Δ are definitely coming within 10% of estimates by high-T series analysis ($\Delta = 1.32 \pm 0.33$) or ε-expansions ($\Delta \cong 1.3$) (Racz and Collins 1976, De Dominicis *et al* 1975).

What we have proposed here is a 'phenomenological' RG approach which operates entirely within the Markoffian dynamics of a nearest-neighbour model and produces neither couplings of longer range in space, nor memory effects of non-zero range in time. Our method is simple and widely applicable as the calculations involve only a small number of degrees of freedom and exploit optimally the physics contained in local fluctuations of the order parameter.

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† Various methods of determination of z in $d = 2$ are discussed in detail by Mazenko and Valls (1981).